ON THE EFFECTIVENESS OF ERROR ESTIMATES OF THE FINITE ELEMENT SOLUTION OF STOCHASTIC PARTIAL DIFFERENTIAL EQUATION

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ABSTRACT

The Stochastic fourth order heat equation driven by a space-time white noise was considered. Error estimates were verified using finite element solvers as a tool for numerical experiments. The proposed solution for the numerical estimate of the strong convergence rate was shown to be effective.

Keywords: Cahn-Hilliard equation, Weak formulation, Space-Time Noise, Finite Element, Error analysis

INTRODUCTION

In recent years, there has been a massive research concerning the numerical approximation of solution of stochastic partial differential equations (SPDEs). For instance, Davis and Gains (2000) considered the numerical solution of SPDE driven by a multiplicative space-time white noise, using finite differences. The authors investigated the extent to which the order of convergence proved by Gyongy (1999) can be improved, and found that better approximations are possible for the case of additive noise ($\sigma(u)$ = constant) if we wish to estimate space averages of the solution rather than point-wise estimates, or if we are permitted to generate other functionals of the noise. But for multiplicative noise, the authors showed that no such improvement is possible.

In this study therefore, we are interested in verifying the theoretical error estimate prove by Njoseh and Ayoola (2008) and Njoseh (2010 and 2013) to show its effectiveness. In those works, they studied the stochastic Cahn-Hilliard equation

$$
\begin{aligned}
&u_t + \Delta^2 u - \Delta f(u) = \sigma(u)W, in\ \Omega \times [0,T] \\
&u(0,\cdot) = u_0\ in\ \Omega \qquad (1.1) \\
&\frac{\partial u}{\partial n} = \frac{\partial \Delta u}{\partial n} = 0\ on\ \partial\Omega \times [0,T]
\end{aligned}
$$

where $u(t)$ is a random process that takes values in $L_2(\Omega)$, Ω is a bounded domain in $\mathbf{R}^d,\ d \le 3$, with a sufficiently smooth boundary $\partial\Omega$. Δ is the Laplacian. W is a standard Brownian motion defined on a filtered probability space $(\Omega, F, \{F_t\}_{t\ge 0}, P)$. f is a locally lipschitz real function and σ is a smooth positive function in $C^4(\Omega)$. (Detailed definitions of these functions and operators can be found in Buckdahn and Pardonx (1990).

Equation (1.1) is a fourth order heat equation used to model a complicated phase separation and Coarsening phenomena in a melted alloy that is quenched to a temperature at which only two different concentration phases can exist stably. This was developed by Cahn and Hilliard in 1958. (For more physical background on this equation, see Novich-Cohen and Segel (1984)). The existence and uniqueness of the solution of (1.1) has been a subject of study for a long time (Da Prato and Zabczyk, 1992; Debussche and Zambotti, 2006). Finite element approximations of the deterministic form of (1.1) was analyzed in the L_2-norms in Elliot and Larsson (1992), and Cardon-Weber (2000) studied the explicit and implicit discretization schemes of (1.1) in dimensions $d \le 3$.

The Finite Element Analysis

We present the finite element method for equation (1.1). We discretize in time by using the backward Euler or implicit Euler method. This is obtained by letting $u^n \in H_0^1$ be the approximation of $u(t,\cdot)$ at time $t = t_n$ and the time derivative $\frac{\partial}{\partial t}u(t,x)$ is approximated by $(u^n - u^{n-1})/k_n$. The time discretized problem is thus to find $u^n := u(t_n,\cdot) \in H_0^1$, such that

$$\int_\Omega u^n(x)v(x)dx + k_n\int_\Omega \nabla^2 u^n(x)\nabla^2 v(x)dx + k_n\int_\Omega \nabla f^n(u(x))\nabla v(x)dx$$
$$= \int_\Omega u^{n-1}(x)v(x)dx + \sum_{j=1}^{J}\gamma_j^{\frac{1}{2}}(\beta_j(t_n) - \beta_j(t_{n-1}))\int_\Omega v(x)\sigma(t_{n-1},x,u^{n-1})e_j(x)dx \quad \forall\, v \in H_0^1 \quad (2.1)$$

where $f^n \in H$ is an approximation of $f(u(t,\cdot))$ at time $t = t_n$. We truncated the sum of the last term to J terms. This is due to the fact that if $\sigma = I$, it is sufficient to take $J = M_h$ (cf Yan (2003a and b)).

Discretizing in space, we seek the approximation in the finite element space V_h instead of in H_0^1. The fully discrete method is then to find $U^n := U(t_n,\cdot) \in V_h$ such that

$$\int_\Omega U^n(x)\chi(x)dx + k_n\int_\Omega \nabla^2 U^n(x)\nabla^2\chi(x)dx + k_n \int_\Omega \nabla f^n(U(x))\nabla\chi(x)dx$$
$$= \int_\Omega U^{n-1}(x)\chi(x)dx + \sum_{j=1}^{J}\gamma_j^{\frac{1}{2}}(\beta_j(t_n) - \beta_j(t_{n-1}))\int_\Omega \chi(x)e_j(x)dx \quad \forall\, \chi \in V_h \qquad (2.2)$$

where we already treated $\sigma = I$. Finally, we write (2.2) in matrix form. Since $U^n \in V_h$, we can write U^n in terms of the basis $\{\phi_k\}_{k=1}^{M_h}$ (where $\phi(P_j) = \begin{cases} 1, & if\ i = j \\ 0, & if\ i \neq j \end{cases}$ and $\{P_j\}_{j=1}^{M_h}$ is the vertices of the triangulations and M_h is the total number of vertices) as

$$U^n = \sum_{i=1}^{M_h}\psi_i^n\phi_i(x) \quad (2.3)$$

where $\psi_i^n = \psi^n(\mathrm{P_i})$. Substituting (2.3) into (2.2) and taking $\chi = \phi_k,\ k = 1,\cdots,M_h,$ our problem can be stated as follows: Find coefficients ψ_i^n, such that

$$\sum_{i=1}^{M_h}\psi_i^n\int_\Omega \Phi_i(x)\Phi_k(x)dx + k_n\sum_{i=1}^{M_h}\psi_i^n\int_\Omega \nabla^2\Phi_i(x)\nabla^2\Phi_k(x)dx + k_n\int_\Omega \nabla f^n(U(x))\nabla\Phi_k(x)dx$$
$$= \sum_{i=1}^{M_h}\psi_i^{n-1}\int_\Omega \Phi_i(x)\Phi_k(x)dx + \sum_{j=1}^{J}\gamma_j^{\frac{1}{2}}(\beta_j(t_n) - \beta_j(t_{n-1}))\int_\Omega \Phi_k(x)e_j(x)dx \qquad (2.4)$$

for $k = 1,\cdots,M_h$. We let φ_j denote the nodal values of the initial approximation u_h^0, $\varphi_j(0) = \varphi_j,\ j = 1,\cdots,M_h$. In matrix, equation (2.4) becomes

$$B\psi^n + k_nA\psi^n + k_nF = B\psi^{n-1} + W^n - W^{n-1}\ for n \geq 0\ and\ \psi^0 = \varphi \qquad (2.5)$$

Here $B = (b_{ik})$ is the mass matrix with elements $b_{ik} = \int_\Omega \Phi_i\Phi_k dx$, $A = (a_{ik})$ is the stiffness matrix with elements $a_{ik} = \int_\Omega \nabla^2\Phi_i\nabla^2\Phi_k dx$, $F = (f_k) = \int_\Omega \nabla f^n(U(x))\nabla\Phi_k(x)dx$, ψ^n is the vector of unknowns ψ_i^n and the vector $W^n - W^{n-1} = (w_k)$ contains the elements

$$w_k = \sum_{j=1}^{J} \gamma_j^{\frac{1}{2}} (\beta_j(t_n) - \beta_j(t_{n-1})) \int_\Omega \Phi_k(x) e_j(x) dx \qquad (2.6)$$

3. Finite element discretization in Space and Time

Let V_h be a family of finite element spaces, where V_h consists of continuous piecewise polynomials of degree $\leq$ 1with respect to the triangulation τ_h of Ω. We shall also assume that $\{V_h\} \subset H_0^1(\Omega)$. According to the standard finite element method, the semi-discrete problem of (1.1) is to find $u_h(t) \in V_h \subset H$, such that,

$$u_{h,t} + A_h^2 u_h + A_h P_h f(u_h) = \sigma(u_h) \partial^2 W,$$
$$t > 0, \quad u_h(0) = u_{0h} \qquad (3.1)$$

with mild solution as

$$\hat{u}_h(t) = E_h(t) u_0 - \int_0^\tau E_h(t-s) A_h P_h f(u(s)) \partial^2 \hat{W} ds$$

Applying the implicit Euler method, for $k = \Delta t,\ t_n = nt,\ \Delta W^n = W(t_n) - W(t_{n-1})$ we have for $U^n \in V_h, U^0 = P_h u_0$ and $\sigma = I;$ we have the fully discrete scheme as

$$\left(\frac{u^n - u^{n-1}}{k}\right) + A_h^2 U^n + A_h P_h f(U^n) = P_h\left(\frac{\hat{W}(t_n) - \hat{W}(t_{n-1})}{k}\right), \quad t_n > 0$$

$$U^n - U^{n-1} + kA_h^2 U^n + kA_h P_h f(U^n) = P_h(\hat{W}(t_n) - \hat{W}(t_{n-1})) \qquad (3.2)$$

and the variation of constants formula for

$$U(t_n) = E(t_n) U^0 - \int_0^{\tau_n} E(t_n - s) A_h P_h f(U^0)) dW(s) \qquad (3.3)$$

becomes (Njoseh and Ayoola (2008))

$$U^n = E_{kh} U^{n-1} - E_{kh} A_h P_h f(U^j) \Delta W^n$$

$$U^n = E_{kh}^n u_{0h} - k \sum_{j=1}^{n} E_{kh}^{n-j+1} A_h P_h f(U^j) \Delta W^j \qquad (3.4)$$

Where $E_{kh} = (1 + kA_h^2)^{-1}$

These finite element discretization in both space and time led to the following theoretical error estimates (Njoseh and Ayoola (2008) and Njoseh (2010 and 2013))

Theorem 1: Let u_h be the spatially semi-discrete approximate solution of order r and with mesh parameter h, and let the initial approximation be chosen as the L_2 - projection of the exact initial value u_0. Then if for $r \leq 2$ and $\left\| A^{\frac{(\gamma-1)}{2}} \right\|_{HS} < \infty$, for $\gamma \in [0,4]$ we have

$$\|u_h(t) - u(t)\|_{L_2} \leq Ch^\gamma \left(\|u_0\|_{L_2(\Omega, \dot{H}^\gamma)} + \left\| A^{\frac{(\gamma-1)}{2}} \right\|_{HS}^2 \right), \quad 0 < t \leq T \qquad (3.5)$$

Theorem 2: Let u be the solution of (1.1) and U^n the fully discrete approximate solution. If $\left\| A^{\frac{(\gamma-1)}{2}} \right\|_{HS} < \infty$, for some $0 \leq \gamma \leq 4$, then

$$\|e_n\|_{L_2(\Omega,H)} = (E(\|U^n - u(t_n)\|^2))^{\frac{1}{2}} \leq C(k^{\frac{\gamma}{2}} + h^\gamma)$$
$$\left(\|u_0\|_{L_2(\Omega, \dot{H}^\gamma)} + \left\| A^{\frac{(\gamma-1)}{2}} \right\|_{HS} \right) \qquad (3.6)$$

If $W(t)$ is a Wiener process, we have

$$\|e_n\|_{L_2(\Omega,H)} \leq C(k^{\frac{\gamma}{2}} + h^\gamma)(1 + \|u_0\|_{L_2(\Omega, \dot{H}^\gamma)}) \qquad (3.7)$$

4. Setup of the Numerical experiments

The main purpose of the numerical experiment is to examine the convergence rate of the numerical method. The numerical experiment is performed on equation (1.1) with the following functions: $T = 1, \sigma \equiv I, f(x) = x^3 - x, u_0(x) = \cos(x)$ where $x = (x_1, x_2) \in \Omega$, Ω is the unit square $\mathbf{R}^2$.

In the numerical experiment, the strong convergence rate in both the spatial and time steps in equation (3.6) is computed. Since the true solution to the SPDE (1.1) itself is a random process, it is not known explicitly. Therefore, the finite element solution computed on a very fine mesh is considered as the true solution, and the finite element solutions computed on the less finer meshes are compared with this numerically obtained "true solution" to compute the strong convergence rate. Due to the lengthy run time of the finite element solver used, we set the fine mesh as $k = 2^{-8}$ and $h = 2^{-8}$ respectively.

5. Analysis of the Strong Convergence rate in k and h

The experimental setup for the strong convergence rate k as described in equation (3.6). We first compute the "true solution" u on the mesh where $h = 2^{-8}$ and $k = 2^{-8}$, which we consider as a fine mesh due to the lengthy run time of the solver. Then, we fix $h = 2^{-8}$ and compute the approximated solution U^k for different time partitions, in particular, for $k = 2^{-7}, 2^{-6}, 2^{-5}, 2^{-4}, 2^{-3}$, respectively. Finally, we compute the $\|U^k - u\|_{L_2(\Omega,H)}$ for every time partition.

Similarly, the strong convergence rate in h is obtained from the numerical experiment after computing the "true solution" u on the fine mesh, we fix $k = 2^{-8}$ and compute the approximate solution U^h on the meshes with $h = 2^{-7}, 2^{-6}, 2^{-5}, 2^{-4}, 2^{-3}$, Then, the error is computed in the same way as for the time step.

Applying Theorem 2, the strong convergence rate is almost $O(k^{\frac{1}{4}})$ and $O(h^{\frac{1}{2}})$, respectively. When estimating the convergence rate in k, we fix h and do the simulations for different k's, and vice versa for the convergence rate in h.

Roughly speaking, we have

$$\|U^n - u(t_n)\|_{L_2(\Omega,H)} := e_{strong}(k,h) \approx C(k^{\frac{1}{4}} + h^{\frac{1}{2}}) \qquad (5.1)$$

Thus, fixing k and h, respectively,

$$\log e_{strong}(k) \approx \log C + \tfrac{1}{4}\log k \qquad (5.2)$$

and

$$\log e_{strong}(h) \approx \log C + \tfrac{1}{2}\log h \qquad (5.3)$$

Hence, one can expect to get a graph with slope close to $\frac{1}{4}$ and $\frac{1}{2}$ on the log-log plot from the numerical experiment, respectively for the strong convergence rate in k and h.

Another means of computing the convergence rate from the obtained computational error data is to show (Bin (2004)) that Theorem 2 implies that the order of strong convergence of our method should be close to $O(k^{\frac{\gamma}{2}} + h^{\gamma})$. If h is sufficiently small, such that the error estimates are dominated by k, the predicted rate of convergence would then be $O(k^{\frac{\gamma}{2}})$. This gives us

$$\frac{U^{k_i}}{U^{k_{i+1}}} \approx \left(\frac{k_i}{k_{i+1}}\right)^{\frac{\gamma}{2}} = 2^{\frac{\gamma}{2}} \qquad (5.4)$$

and from that we obtain

$$\gamma = \frac{2}{\log 2}\log\left(\frac{U^{k_i}}{U^{k_{i+1}}}\right) \qquad (5.5)$$

In the same way, when k is very small, the error is assumed to be dominated by h and the rate of convergence should be $O(h)$. Similarly to (5.5), we obtain,

$$\gamma = \frac{1}{\log 2}\log\left(\frac{U^{h_i}}{U^{h_{i+1}}}\right) \qquad (5.6)$$

Therefore, using (5.5) and (5.6), we obtain the results for γ as shown in table (5.1). The table shows that the average of the γ's is around the expected value $\frac{1}{2}$.

Table 5.1: Convergence rate in k and h

Hence we can conclude that the proposed finite element solution for the numerical estimate of the strong convergence rate

h	K	Γ	K	h	Γ
2^{-8}	2^{-7}	0.7658	2^{-8}	2^{-7}	0.8109
2^{-8}	2^{-6}	0.5624	2^{-8}	2^{-6}	0.4075
2^{-8}	2^{-5}	0.4456	2^{-8}	2^{-5}	0.3816
2^{-8}	2^{-4}	0.7658	2^{-8}	2^{-4}	0.4397

will prove to be effective.

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